**Issues with Using Harmonic Means in Business Valuation**

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**Abstract:** The harmonic mean is a measure of central tendency for a distribution or data set. It is used frequently for ratios, such as price/earnings (P/E) ratios in financial and valuation analysis. Use of the harmonic mean is appropriate when small fractional observationsare not present, but is not resistant to small ratios. In this paper, we illustrate the need for caution when using harmonic means using symmetric and skewed distributions.

**Introduction**

In the valuation of corporate entities, analysts commonly calculate and evaluate financial ratios such as the stock price/earnings (P/E), earnings before interest and taxes (EBIT), and market value to invested capital (MVIC), among others. When comparing such ratios to a peer group of related companies, it is often necessary to calculate a “typical” comparison ratio, or a measure representing the “center” of the related peer firm portfolio. Frequently used measures of center include the arithmetic mean (or simply “the mean”), median, mode, geometric mean, and harmonic mean. For financial ratio analysis, the harmonic mean is considered the most appropriate measure.

In fact, Baker and Ruback (1999) argue that the harmonic mean is always the most appropriate measure of central tendency for financial ratio analysis. Part of the appeal of the harmonic mean is its interpretation with respect to ratios and their comparisons. For example, for stock price divided by earnings per share (P/E) ratios, harmonic means give an equal weight to each dollar invested in a portfolio rather than to each instrument (firm) – which is a desirable feature when dealing with prices at different scales. However, when using harmonic means for firm valuations, we must also consider other properties of this measure. In fact, because of certain potential pitfalls, some have argued against using this particular measure of central tendency.[[2]](#footnote-2) In this paper, we discuss several aspects of the harmonic mean, and recommend caution when using it to estimate the center of certain distributions of ratios.

**Arithmetic Means and their Properties**

If we transform data (i.e., let y\* = a + by), the arithmetic mean (ȳ) for the transformed data is simply computed by analogously computing ȳ\* = a + bȳ. This property does not hold for harmonic means. It holds for multiplicative transformations with no additive constant, but does not hold for transformations with additive constants with or without multiplicative constants.

Define H = , where H = harmonic mean; n = number of observations; y = an observed value of the variable of interest. Note, if we let y\* = by, H\* = bH since

H\* = = = bH.

A simple example illustrates that the additive component does not follow analogously. Consider the observations 1 and 2. The harmonic mean is H = 1.333 (or 4/3). If we add 1 to each number (i.e., let y\* = y + 1), our transformed observations have harmonic mean H\* = 2.4 (or 12/5). Thus, H\* ≠ H + 1. Another issue with additive constants is that harmonic means exists for data sets containing only positive values. The result is that a given quantity is subtracted from each observation - if this causes at least one transformed observation to be negative or zero - the harmonic mean for the transformed data will not exist.

An important drawback when using the arithmetic mean is that it is not resistant to outliers (unusually large or small values in the data set). The harmonic mean also is not resistant to outliers, but only for small values. Additionally, the definition of “outlier” must be carefully considered for harmonic means. For example, consider the three data sets: S1 = {1, 2, 3, 4, 5}; S2 = {1, 2, 3, 4, 20}; S3 = {0.1, 1, 2, 3, 4}. The resulting harmonic means are H1 = 300/137 (or 2.19), H2 = 75/32 (or 2.34), and H3 = 60/145 (or 0.41). Note that the harmonic mean increases by 0.15 (2.34 – 2.19) when the value 5 in S1 is changed to a 20 in S2. When the value 5 in S1 is changed to a 0.1 in S3, the harmonic mean decreases by 1.78 (2.19 – 0.41).

The change is far more dramatic when the “outlier” is a small value. Furthermore, most outlier-detection techniques would tag the value 20 in S2, while the value 0.1 in S3 would go undetected. The reason for this is that 0.1 is far closer to the value 1 (the closest value to it in S1) than is 20 to the value 4 (which is its closest neighbor in S3). In fact, note that S3 is very similar to values in S1 transformed by subtracting 1 from each. (If the 0.1 had been a 0, this would have been the case. However, the harmonic mean would not be defined for a data set including a value of 0.)

While these examples are somewhat instructive, we now consider a more theoretical approach. In the next section, we use two well-known and used distributions from simulation – the uniform and triangular – to show the effects of skewness and fractional observations on harmonic means.

**Harmonic Means: Considerations for Uniform and Triangular Distributions**

Uniform distributions are perfectly symmetric, with equal probabilities given to any similar range of observations that are possible. A uniform distribution is defined by the probability density function (pdf)

f() =

and cumulative density function (CDF)

F() =

(See, e.g., Johnson and Kotz 1995.) An example of a uniform distribution (with parameters a = 1 and b = 3) is given in Figure 1.

Triangular distributions may be symmetric or skewed (with varying degrees of skewness). A triangular distribution is defined by the pdf

f()

and CDF

F() =

Note that c is the modal value. If c is equidistant from both a and b, the distribution is symmetric. Otherwise, there will be varying degrees of skewness depending on the relative distances from a and b. In the most extreme cases, c is equal to either a or b. (See, e.g., Johnson et al 1995.) Examples of triangular distributions are given in Figures 2 and 3. In Figure 2, c = 2, and the distribution is symmetric. In Figure 3, c=1, and the distribution is heavily right-skewed.

Analyzing data from uniform and triangular distributions offers insights into use of the harmonic mean with distributions having various shapes and ranges of possible values. In Tables 1-6, we summarize information from twenty-four distributions of six types: four uniform distributions, and twenty triangular distributions. We considered a mix of triangular distributions, as they have varying degrees of skewness. The distributions under each type category differ with respect to the probability of encountering values between zero and one (the fractional values that tend to strongly affect harmonic means).

In each table, four distributions are summarized and are given: appropriate parameter values (a and b for uniform distributions; a, b, and c for triangular distributions), skewness coefficients, the arithmetic means, the harmonic means, and the percentiles of the harmonic means and arithmetic means. The rationale behind reporting percentile values is that a value representative of the “center” of a distribution would be expected to have a percentile value somewhat close to 50 (the percentile of the median).

In Table 1, four uniform distributions are summarized. Each uniform distribution has a range (b-a) of 5. Note that since all uniform distributions are symmetric (skewness = 0), the arithmetic mean (µ) is also the median (i.e., always falls at the 50th percentile). What is noteworthy in the table is that the harmonic mean percentile plummets from 35.81 when a = 1 to 15.89 when a = 0.01. The presence of fractional values causes the harmonic mean to be extremely small relative to the rest of the distributional values.

In each of the Tables 2-6, we present summaries of four triangular distributions with a range (b-a) of 5 (chosen for consistency with the uniform distribution ranges). Unlike the uniform distributions, triangular distributions may be symmetric or skewed. Tables 2 and 4 summarize right-skewed triangular distributions, and Tables 3 and 4 contain left-skewed distributions. In Table 6, the triangular distributions are perfectly symmetric.

In each case, the percentiles of the harmonic means decline as more values fall between zero and one. The most dramatic case occurs in Table 2, in which the distribution is heavily right-skewed (a = c). This is not unexpected, since the prevalence of fractional values is highest among the distributions under consideration. What is perhaps more surprising is that a similar effect is seen with the uniform distributions. In any event, *every* harmonic mean observed has a percentile value below 40 when the corresponding distribution contains observations between zero and one. When ratios below 0.5 can be observed, only one harmonic mean (from Table 3) has a percentile value above 30.

**Effects on Valuation Analysis**

Data for company ratios of interest to valuation analysts tend to be right-skewed, as many firms have low values and fewer and fewer have relatively high ones. (See Figure 1 for an example of PE ratio data.) As we observed in the right-skewed triangular distributions, the presence of values between zero and one results in percentile values falling far from 50.

Valuation analysts should take care in using harmonic means as a default measure of central tendency. This is not to suggest that use of the harmonic mean should be eschewed. Rather, one must be diligent with respect to checking the empirical distributions of measures. When a distribution is prone to have fractional values (especially values close to zero), these values will have a profound impact on the harmonic mean. In these cases, the harmonic mean is not representative of a “typical” value. Justifying it as such likely would prove quite difficult to a discerning audience.

Figure 1. Uniform (1, 3) distribution

Figure 2. Symmetric triangular (1, 2, 3) distribution

Figure 3. Extremely right-skewed triangular (1, 1, 3) distribution

Figure 4. PE ratio example plot

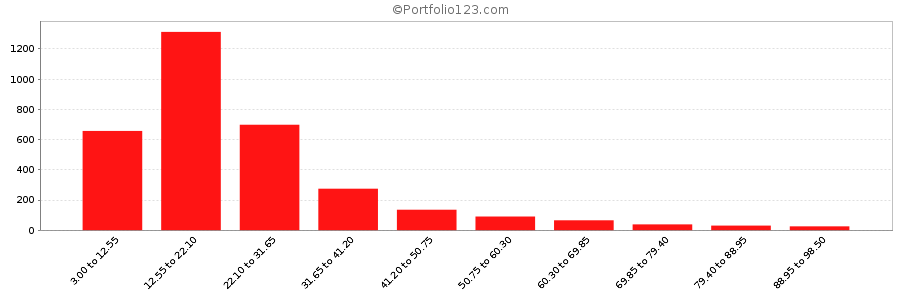


Table 1. Uniform distributions

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Uniform distributions (b – a = 5) | | | | |
| b, a | 6, 1 | 5.5, 0.5 | 5.1, 0.1 | 5.01, 0.01 |
| Skewness | 0 | 0 | 0 | 0 |
| µ | 3.50 | 3.00 | 2.60 | 2.51 |
| H | 2.79 | 2.09 | 1.27 | 0.80 |
| Percentile of µ | 50.00 | 50.00 | 50.00 | 50.00 |
| Percentile of H | 35.81 | 31.70 | 23.43 | 15.89 |

Table 2. Most extreme right-skewed triangular distributions

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Triangular distributions (b – a = 5) | | | | |
| b, c, a | 6, 1, 1 | 5.5, 0.5, 0.5 | 5.1, 0.1, 0.1 | 5.01, 0.01, 0.01 |
| Skewness | +0.57 | +0.57 | +0.57 | +0.57 |
| µ | 2.67 | 2.17 | 1.77 | 1.68 |
| H | 2.17 | 1.53 | 0.83 | 0.48 |
| Percentile of µ | 55.56 | 55.56 | 55.56 | 55.56 |
| Percentile of H | 41.44 | 36.85 | 27.08 | 17.85 |

Table 3. Less extreme right-skewed triangular distributions

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Triangular distributions (b – a = 5) | | | | |
| b, c, a | 6, 2.25, 1 | 5.50, 1.75, 0.50 | 5.10, 1.35, 0.10 | 5.01, 1.26, 0.01 |
| Skewness | +0.42 | +0.42 | +0.42 | +0.42 |
| µ | 3.08 | 2.58 | 2.18 | 2.09 |
| H | 2.72 | 2.12 | 1.56 | 1.38 |
| Percentile of µ | 54.63 | 54.63 | 54.63 | 54.63 |
| Percentile of H | 42.47 | 39.12 | 33.28 | 29.91 |

Table 4. Most extreme left-skewed triangular distributions

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Triangular distributions (b – a = 5) | | | | |
| b, c, a | 6, 6, 1 | 5.5, 5.5, 0.5 | 5.1, 5.1, 0.1 | 5.01, 5.01, 0.01 |
| Skewness | -0.57 | -0.57 | -0.57 | -0.57 |
| µ | 4.33 | 3.83 | 3.43 | 3.34 |
| H | 3.90 | 3.29 | 2.71 | 2.53 |
| Percentile of µ | 44.44 | 44.44 | 44.44 | 44.44 |
| Percentile of H | 33.55 | 31.10 | 27.32 | 25.43 |

Table 5. Less extreme left-skewed triangular distributions

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Triangular distributions (b – a = 5) | | | | |
| b, c, a | 6, 4.75, 1 | 5.50, 4.25, 0.50 | 5.10, 3.85, 0.10 | 5.01, 3.76, 0.01 |
| Skewness | -0.42 | -0.42 | -0.42 | -0.42 |
| µ | 3.92 | 3.42 | 3.02 | 2.93 |
| H | 3.54 | 2.94 | 2.38 | 2.20 |
| Percentile of µ | 45.37 | 45.37 | 45.37 | 45.37 |
| Percentile of H | 34.46 | 31.86 | 27.76 | 25.66 |

Table 6. Symmetric triangular distributions

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Triangular distributions (b – a = 5) | | | | |
| b, c, a | 6,3.5, 1 | 5.5, 3.0, 0.5 | 5.1, 2.6, 0.1 | 5.01, 2.51, 0.01 |
| Skewness | 0 | 0 | 0 | 0 |
| µ | 3.50 | 3.00 | 2.60 | 2.51 |
| H | 3.15 | 2.56 | 2.01 | 1.83 |
| Percentile of µ | 50.00 | 50.00 | 50.00 | 50.00 |
| Percentile of H | 37.14 | 34.07 | 29.17 | 26.62 |

**Appendix**

The derivations of the harmonic mean formulas are given in this appendix.

The harmonic mean of a distribution is H = 1/(E(1/), where

E(1/) = .

(See, e.g., Mood et al, 1974.)

For uniform distributions,

E(1/) = = [ln() – ln()],

and thus

H = ( -)/[ln() – ln()],

where ln(y) = the natural log of y.

For triangular distributions,

E(1/ E(1/)) = +

= {() + + {(

= { - },

And thus

H = { - }-1

Note that when = ,

E(1/) = { - },

and

H = { - }-1.

Similarly, when b = c,

E(1/) = {},

and

H = { }-1.

**References**

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Johnson, N. L., Kotz, S., & Balakrishnan, N. (1995). *Continuous Univariate Distributions. Vol. 1*. New York: Wiley.

Mood, A. M., Graybill, F. A., & Boes, D. C. (1974). *Introduction to the Theory of Statistics*, New York: McGraw-Hill.

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2. For example, see Dohmeyer, Kierulff and Castell, Business Valuation Update, Jan. 2015. [↑](#footnote-ref-2)